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THE DIFFRACTION OF ELASTIC WAVES BY SPHERICAL DEFECTS†

O. A. NAZARENKO and G. Ya. POPOV

Odessa

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A method of reducing a number of diffraction problems to a system of one-dimensional integro-differential equations is proposed based on the method of discontinuous solutions [1, 2] in the case of steady elastic waves. The defect can be either a spherical crack or a thin rigid spherical inclusion. The method is described in detail for the second case. An effective approximate method of solving the corresponding integro-differential equation in the class of functions with non-integrable singularities is proposed in the case of the diffraction of a torsional wave. A numerical realization of the method is given, namely, graphs of the reactive torsional moment (the inclusion is rigidly fixed) as a function of the oscillation frequency and dimensions of the inclusion are drawn, and the same graphs for the amplitude of the torsional oscillations of the inclusion when it is mobile (not fixed).

1. CONSTRUCTION OF A DISCONTINUOUS SOLUTION OF THE WAVE EQUATION FOR A SPHERICAL DEFECT

We mean by a defect [1, 2] part of a surface, on intersecting which the displacements and stresses undergo discontinuities of the first kind. A typical defect is a mathematical cut on a certain part of the surface (a crack). Another case of a defect is a thin rigid inclusion in the form of a shell, the median surface of which coincides with the same part of the surface. Here we are considering the case when the defect is part of a spherical surface: $r = R$, $0 \leqslant \theta \leqslant \omega$, $-\pi \leqslant \varphi \leqslant \pi$ where r, θ, φ are spherical coordinates. We know [3, 4] that the solution of the equations of motion of an elastic isotropic medium can be expressed in terms of wave functions. Hence, before constructing the discontinuous solution of the equations of motion we will construct such a solution for the wave equation

$$\Delta\Phi - c^{-2}\frac{\partial^2}{\partial t^2}\Phi = 0,\ \ 0<r<\infty,\ \ 0<\theta<\pi,\ |\varphi|<\pi,\ \ t\geqslant 0 \tag{1.1}$$

where Δ is the Laplace operator in a spherical system of coordinates.

We mean by a discontinuous solution of Eq. (1.1), specified over the whole of space, for a spherical defect

$$r=R,\ \ 0\leqslant\theta\leqslant\pi,\ \ -\pi\leqslant\varphi\leqslant\pi \tag{1.2}$$

the solution of Eq. (1.1) which satisfies it everywhere, apart from the points of the defect (1.2). At these points the function and its normal derivative (to the surface of the defect) undergo discontinuities of the first kind and their sudden changes are specified, for which we introduce the following notation

$$\Phi(R-0,\theta,\varphi,t)-\Phi(R+0,\theta,\varphi,t)=\langle\Phi\rangle$$
$$\Phi'(R-0,\theta,\varphi,t)-\Phi'(R+0,\theta,\varphi,t)=\langle\Phi'\rangle$$

Here and everywhere below the derivatives with respect to r will be denoted by a prime, the derivative with respect to θ will be denoted by a dot, and the derivative with respect to φ will be denoted by a comma. To construct this solution we will use the same scheme as in [1, 2]. Applying integral and Fourier transformations

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$$\Phi_p = \int_0^\infty \Phi(r,\theta,\varphi,t)e^{-pt}dt, \quad \Phi_{pn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_p(r,\theta,\varphi)e^{-in\varphi}d\varphi \tag{1.3}$$

and a Legendre transformation ($P_k^m(\cos\theta)$ is the associated Legendre polynomial)

$$\Phi_{pnk}(r) = \int_0^\pi \sin\theta P_k^{|n|}(\cos\theta)d\theta \tag{1.4}$$

in succession to Eq. (1.1) we can reduce this equation to the following one-dimensional equation

$$r^{-2}\left[(r^2\Phi'_{pnk})' - k(k+1)\Phi_{pnk}\right] - p^2c^{-2}\ \Phi_{pnk} = 0, \ 0 < r < \infty \tag{1.5}$$

It is required to construct a discontinuous solution of this equation with specified abrupt changes

$$\begin{aligned} \left\langle\Phi_{pnk}\right\rangle &= \Phi_{pnk}(R-0) - \Phi_{pnk}(R+0) \\ \left\langle\Phi'_{pnk}\right\rangle &= \Phi'_{pnk}(R-0) - \Phi'_{pnk}(R+0) \end{aligned} \tag{1.6}$$

To do this, we reduce Eq. (1.5) to the Bessel equation by making the substitution $\varphi_{pnk}(r) = r^{1/2}\Phi_{pnk}(r)$. We then apply an integral Hankel transformation

$$\varphi_{pnk\alpha} = \int_0^\infty rJ_{k+1/2}(\alpha r)\varphi_{pnk}(r)dr$$

to this using the generalized scheme in [1]. We thereby obtain the Hankel transformant of the discontinuous solution of Eq. (1.5) with abrupt changes (1.6). Then applying the inversion formula for the Hankel transformation and also formula 6.541(1) from [1] from 5, we obtain the required discontinuous solution of Eq. (1.5) with abrupt changes (1.6)

$$\Phi_{pnk}(r) = R^2\left[\left\langle\Phi'_{pnk}\right\rangle\Gamma_k(r,R) - \left\langle\Phi'_{pnk}\right\rangle\frac{\partial}{\partial R}\Gamma_k(r,R)\right]$$

$$\Gamma_k(r,R) = \frac{1}{\sqrt{rR}}\begin{cases} I_\nu\left(\dfrac{Rp}{c}\right)K_\nu\left(\dfrac{rp}{c}\right), & r > R \\ I_\nu\left(\dfrac{rp}{c}\right)K_\nu\left(\dfrac{Rp}{c}\right), & r < R \end{cases} \tag{1.7}$$

$$\nu = k + 1/2, \ k = 0,1,2,\ldots$$

($I_\nu(z)$ is the modified Bessel function and $K_\nu(z)$ is the MacDonald function). In order to obtain a discontinuous solution of the initial wave equation, we must use the inversion formulae for the Legendre transformant [6]

$$\begin{aligned} \Phi_{pn}(r,\theta) &= \sum_{k=|n|}^{\infty} \Phi_{pnk}(r)\sigma_{kn}P_k^{|n|}(\cos\theta) \\ \sigma_{kn} &= (k-|n|)!(2k+1)[2(k+|n|)!]^{-1} \end{aligned} \tag{1.8}$$

and also for the Fourier and Laplace transformants.

For example, using (1.8) instead of (1.7), we obtain

$$\Phi_{pn}(r,\theta) = R^2\left[\int_0^\omega \left\langle\Phi'_{pn}\right\rangle K_n(\theta,\tau;r,R)\sin\tau d\tau - \right.$$

$$\left. - \int_0^\omega \left\langle\Phi_{pn}\right\rangle \sin\tau\frac{\partial}{\partial R}K_n(\theta,\tau;r,R)d\tau\right] \tag{1.9}$$

$$K_n(\theta,\tau,r,R) = \sum_{k=|n|}^{\infty} \frac{P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\tau)}{[\sigma_{kn}\Gamma_k(r,R)]^{-1}}$$

If the oscillation process described by wave equation (1.1) is steady, i.e. it occurs harmonically

$$\Phi(r,\theta,\varphi,t) = e^{-i\omega_0 t}\tilde{\Phi}(r,\theta,\varphi) \tag{1.10}$$

to eliminate the time there is no need to employ the Laplace transformation, and for the functions $\tilde{\Phi}(r, \theta, \varphi)$ we obtain Eq. (1.5), in which we must put $p = -i\omega_0$.

The discontinuous solution in this case, instead of (1.7), will have the form

$$\tilde{\Phi}_{nk}(r) = R^2\left[\langle\Phi'_{nk}\rangle\Gamma_{d,k}(r,R) - \langle\Phi_{nk}\rangle\frac{\partial}{\partial R}\Gamma_{d,k}(r,R)\right]$$
$$\Gamma_{d,k}(r,R) = \frac{\pi i}{2\sqrt{rR}}\begin{cases} J_\nu(Rd)H_\nu^{(1)}(rd), & r > R \\ J_\nu(rd)H_\nu^{(1)}(Rd), & r < R \end{cases} \tag{1.11}$$
$$2\nu = 2k+1, \quad d = c^{-1}\omega_0$$

or, after inverting the Legendre transformant

$$\tilde{\Phi}_n(r,\theta) = R^2\left[\int_0^\omega \langle\tilde{\Phi}'_n\rangle \sin\tau K_d^n(\theta,\tau;r,R)d\tau - \right.$$
$$\left. -\int_0^\omega \langle\tilde{\Phi}_n\rangle \sin\tau \frac{\partial}{\partial R} K_d^n(\theta,\tau;r,R)d\tau\right] \tag{1.12}$$
$$K_d^n(\theta,\tau;r,R) = \sum_{k=|n|}^{\infty} \sigma_{kn}\Gamma_{d,k}(r,R)P_k^{|n|}(\cos\theta)P_k^{|n|}(\cos\tau)$$

In order to satisfy the radiation condition at infinity, when substituting $p = -i\omega_0$ into (1.7) we chose the first Hankel function $H_\nu^{(1)}(x)$.

When using discontinuous solutions (1.9) and (1.12) in particular problems, an integral representation is required for the function

$$B_k(z) = I_\nu(z)K_\nu(z)\Big|_{z=-i\xi} = \frac{\pi i H_\nu^{(1)}\xi}{2J_\nu^{-1}(\xi)} = A_k(\xi), \quad \nu = k + \tfrac{1}{2} \tag{1.13}$$

to obtain which it is sufficient to use formula 5.9.2(14) from [7], which gives an expansion of the function $\Omega_0(\theta) = I_0(\theta) - L_0(\theta)$ ($L_0(\theta)$ is the second Struve function [5]), in series in an orthogonal system of functions $\cos[(k + 1/2)\theta]$ and therefore

$$B_k(z) = \frac{(-1)^k}{2}\int_0^\pi \Omega_0\left(2z\cos\frac{\theta}{2}\right)\cos\left[\left(k+\frac{1}{2}\right)\theta\right]d\theta$$

By integration by parts using (1.13) we establish

$$A_k(\xi) = \frac{1-\Delta_k(\xi)}{2k+1}, \quad \Delta_k(\xi) = \int_0^\pi \frac{\sin[(k+\frac{1}{2})\theta]}{(-1)^k}\frac{\partial}{\partial\theta}O_0\left(2\xi\cos\frac{\theta}{2}\right)d\theta \tag{1.14}$$

where $O_0(\theta) = J_\nu(\theta) - iH_0(\theta)$, $H_0(\theta)$ is the first Struve function.

2. CONSTRUCTION OF A DISCONTINUOUS SOLUTION OF THE EQUATIONS OF MOTION OF AN ELASTIC MEDIUM FOR A SPHERICAL DEFECT

We will use the well-known solution of the equations of motion of an elastic isotropic medium [3], expressed in terms of three wave functions $\Phi(r, \theta, \varphi, t)$, $\Psi_j(r, \theta, \varphi, t)$, $(j = 1, 2)$ where the first function gives the expansion wave and satisfies wave equation (1.1), in which $c = c_1$, and the other two give the shear wave, and we must put $c = c_2$ in (1.1) where c_1 and c_2 are the velocities of the expansion and shear waves, respectively. If we have in mind steady harmonic oscillations with frequency ω_0 ($[\Phi, \Psi_j] = e^{-i\omega_0 t}[\tilde{\Phi}, \tilde{\Psi}_j]$), changes to Fourier transformants

$$\left[\Phi_n, \Psi_{jn}, u_{rn}, u_{\theta n}, u_{\varphi n}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\varphi}[\Phi, \Psi_j, u_r, u_\theta, u_\varphi]d\varphi, \quad n = 0, \pm 1, \pm 2, \ldots \tag{2.1}$$

and omit the time factor $\exp(-i\omega_0 t)$ and the tilde over the symbols everywhere, the solution can be written in the form

$$\begin{aligned} u_{rn} &= \Phi_n' - (r\sin\theta)^{-1}[\sin\theta\,\Psi_{2n}]^{\cdot} - (r\sin^2\theta)^{-1}n^2\Psi_{2n} \equiv u_n \\ u_{\theta n} &= r^{-1}\Phi_n^{\cdot} + r^{-1}(r\Psi_{2n}^{\cdot})' + in(\sin\theta)^{-1}\Psi_{1n} \equiv \upsilon_n \\ u_{\varphi n} &= (r\sin\theta)^{-1}in\Phi_n + (r\sin\theta)^{-1}in(r\Psi_{2n})' - \Psi_{1n}^{\cdot} \equiv w_n \end{aligned} \tag{2.2}$$

Here the functions Φ_n and Ψ_{jn} will satisfy the Helmholtz equations

$$r^{-2}\left[(r^2[\Phi_n', \Psi_{jn}'])' - \nabla_n[\Phi_n, \Psi_{jn}]\right] + [a^2\Phi_n, b^2\Psi_{jn}] = 0 \tag{2.3}$$

$$\nabla n f(r,\theta) \equiv n^2\sin^{-2}\theta f(r,\theta) - \sin^{-1}\theta[\sin\theta f^{\cdot}(r,\theta)]^{\cdot} \tag{2.4}$$

$$a = \omega_0 c_1^{-1}, \quad b = \omega_0 c_2^{-1}$$

Using Hooke's law and the Cauchy relations, we obtain from the transformants of the displacements (2.2) the transformants of the stresses [3]

$$\begin{aligned} (2\mu)^{-1}\sigma_{rn} &= \Phi_n'' - \lambda(2\mu)^{-1}a^2\Phi_n + b^2\Psi_{2n} + b^2 r\Psi_{2n}' + 3\Psi_{2n}'' + r\Psi_{2n}''' \\ (2\mu)^{-1}\tau_{rn} &= r^{-1}\Phi_n'^{\cdot} - r^{-2}\Phi_n^{\cdot} - \tfrac{1}{2}in\,r(\sin\theta)^{-1}(r^{-1}\Psi_{1n})' + \Psi_{2n}''^{\cdot} + r^{-1}\Psi_{2n}'^{\cdot} - r^{-2}\Psi_{2n}^{\cdot} + \tfrac{1}{2}b^2\Psi_{2n}^{\cdot} \\ -(2\mu)^{-1}\tau_{\varphi n} &= in(r\sin\theta)^{-1}[\Phi_n' - r^{-1}\Phi_n] + \tfrac{1}{2}r(r^{-1}\Psi_{1n}^{\cdot})' + \\ &+ in(\sin\theta)^{-1}\left[\Psi_{2n}'' + r^{-1}\Psi_{2n}' - r^{-2}\Psi_{2n} + \tfrac{1}{2}b^2\Psi_{2n}\right] \end{aligned} \tag{2.5}$$

where μ, λ are the Lamé parameters, and $\tau_{\theta n}$ and $\tau_{\varphi n}$ are the Fourier transformants of the shear stresses $\tau_{r\theta}$ and $\tau_{r\varphi}$, respectively.

We will mean by a discontinuous solution of the equations of motion of an elastic medium for a spherical defect (1.2) the solution of the equations which satisfies these equations everywhere, apart from the points of the defect. At these points the components of the displacement and stress field undergo discontinuities of the first kind with specified jumps. We will introduce the following notation for the Fourier transformants of these jumps

$$\langle u_n\rangle = u_{rn}(R-0,\theta) - u_{rn}(R+0,\theta), \ \langle\upsilon_n\rangle, \langle w_n\rangle, \langle\sigma_{rn}\rangle, \langle\tau_{\theta n}\rangle, \langle\tau_{\varphi n}\rangle \tag{2.6}$$

This discontinuous solution will be constructed using the scheme described in [1, 2]. Here it is more convenient to introduce the following notation instead of v_n, w_n and $\tau_{\theta n}$, $\tau_{\varphi n}$

$$\begin{aligned} \sin\theta\, z_n(r,\theta) &= [\sin\theta\,\upsilon_\nu(r,\theta)]^{\cdot} - in w_n(r,\theta) \\ \sin\theta\, z_n^*(r,\theta) &= [\sin\theta\, w_n(r,\theta)]^{\cdot} + in\upsilon_n(r,\theta) \\ \sin\theta\,\tau_n(r,\theta) &= [\sin\theta\,\tau_{\theta n}(r,\theta)]^{\cdot} - in\tau_{\varphi n}(r,\theta) \\ \sin\theta\,\tau_n^*(r,\theta) &= [\sin\theta\,\tau_{\varphi n}(r,\theta)]^{\cdot} + in\tau_{\theta n}(r,\theta) \end{aligned} \tag{2.7}$$

Taking (2.4) into account we can obtain the following relations from (2.2)

$$u_n(r,\theta) = \Phi_n'(r,\theta) + r^{-1}\nabla_n\Psi_{2n}(r,\theta)$$
$$z_n^*(r,\theta) = \nabla_n\Psi_{1n}(r,\theta); \quad rz_n(r,\theta) = -\nabla_n\left[\Phi_n(r,\theta) + (r\Psi_{2n}(r,\theta))'\right] \tag{2.8}$$

According to [1, 2], it is necessary to express the jumps in the functions $\langle\Phi_n\rangle$, $\langle\Phi_n'\rangle$, $\langle\Psi_{jn}\rangle$, $\langle\Psi_{jn}'\rangle$ in terms of the specified jumps (2.6) or the jumps of the functions (2.7). Changing in (2.2) to jumps and carrying out the necessary combinations in order to obtain the jumps of the functions (2.7), we obtain the relations

$$\langle u_n\rangle = \langle\Phi_n'\rangle + R^{-1}\nabla_n\langle\Psi_{2n}\rangle, \quad \langle z_n^*\rangle = \nabla_n\langle\Psi_{1n}\rangle$$
$$R\langle z_n\rangle = -\nabla_n\left[\langle\Psi_n\rangle + \langle\Psi_{2n}\rangle + R\langle\Psi_{2n}'\rangle\right] \tag{2.9}$$

In order to obtain similar relations for the stresses, in (2.5) we must eliminate terms containing derivatives with respect to r higher than the first order. To do this we use Eqs (2.3), which enables us to write

$$\Phi_n'' = r^{-2}\nabla_n\Phi_n - 2r^{-1}\Phi_n' - a^2\Phi_n$$
$$\Psi_{jn}'' = r^{-2}\nabla_n\Psi_{jn} - 2r^{-1}\Psi_{jn}' - b^2\Psi_{jn}, \quad j = 1,2$$

Using these formulae to eliminate the derivatives from (2.5) and changing to jumps in the stresses (2.6), including also the jumps in τ_n^* and τ_n, we obtain

$$(2\mu)^{-1}\langle\sigma_m\rangle R^2 = \nabla_n\langle\Phi_n\rangle - 2R\langle\Phi_n'\rangle - \tfrac{1}{2}b^2R^2\langle\Phi_n\rangle + R\nabla_n\langle\Psi_{2n}'\rangle - \nabla_n\langle\Psi_{2n}\rangle$$
$$R\mu^{-1}\langle\tau_n^*\rangle = \nabla_n[R\langle\Psi_{1n}'\rangle - \langle\Psi_{1n}\rangle]$$
$$2\mu^{-1}\langle\tau_n\rangle R^2 = -\nabla_n\left\{R\langle\Phi_n'\rangle - \langle\Phi_n\rangle + \nabla_n\langle\Psi_{2n}\rangle - \tfrac{1}{2}b^2\langle\Psi_{2n}\rangle - R\langle\Psi_{2n}'\rangle - \langle\Psi_{2n}\rangle\right\} \tag{2.10}$$

In order to invert relations (2.9) and (2.10), i.e. to obtain from them the jumps in the wave functions, we apply the integral Legendre transformation (1.4) to them, and, after some reduction, we obtain

$$k(k+1)\langle\Psi_{1nk}\rangle = \langle z_{nk}^*\rangle, \quad k(k+1)\langle\Psi_{1nk}'\rangle = R^{-1}\langle z_{nk}^*\rangle + \mu^{-1}\langle\tau_{nk}^*\rangle$$
$$k(k+1)b^2R\langle\Psi_{2nk}\rangle = \mu^{-1}R\langle\tau_{nk}\rangle + 2k(k+1)\langle u_{nk}\rangle + 2\langle z_{nk}\rangle$$
$$-Rb^2\langle\Phi_{nk}\rangle = 4\langle u_{nk}\rangle + 2\langle z_{nk}\rangle + \mu^{-1}R\langle\sigma_{rnk}\rangle$$
$$R^2b^2\langle\Phi_{nk}'\rangle = [b^2R^2 - 2k(k+1)]\langle u_{nk}\rangle - \mu^{-1}R\langle\tau_{nk}\rangle - 2\langle z_{nk}\rangle \tag{2.11}$$
$$k(k+1)b^2R^2\langle\Psi_{2nk}'\rangle = 2\langle u_{nk}\rangle k(k+1) - \mu^{-1}R\langle\tau_{nk}\rangle +$$
$$+\langle z_{nk}\rangle[2k(k+1) - 2 - b^2R^2] + \mu^{-1}Rk(k+1)\langle\sigma_{rnk}\rangle$$

By (1.11) the Fourier–Legendre transformants of the wave functions can be expressed by the formulae

$$\Phi_{nk}(r) = R^2[\langle\Phi_{nk}'\rangle\Gamma_{a,k}(r,R) - \langle\Phi_{nk}\rangle\frac{\partial}{\partial R}\Gamma_{a,k}(r,R)]$$
$$\Psi_{jnk}(r) = R^2\left[\langle\Psi_{jnk}'\rangle\Gamma_{b,k}(r,R) - \langle\Psi_{jnk}\rangle\frac{\partial}{\partial R}\Gamma_{b,k}(r,R)\right], \quad j = 1,2 \tag{2.12}$$

Substituting the values of the jumps (2.11) here and then inverting the Legendre transformation, using (1.8) we obtain the functions Φ_n and Ψ_n, and from them, using (2.2) and (2.5), we obtain the required discontinuous solution of the equations of motion for a spherical defect (1.2). Having the discontinuous solution, we can easily reduce the problem of diffraction by such a defect to one-dimensional integral or integro-differential equations. We will carry out the appropriate operations as they apply to a defect in the form of a thin spherical inclusion.

3. REDUCTION OF THE PROBLEM OF THE DIFFRACTION OF AN ELASTIC WAVE BY A THIN SPHERICAL INCLUSION TO THE INTEGRO-DIFFERENTIAL EQUATIONS

Suppose the defect is a thin inclusion in the form of a thin rigid spherical shell, the median surface of which is fixed by relations (1.2). We will assume that the inclusion is perfectly bonded to the elastic medium. Suppose a steady elastic wave is incident on this inclusion and causes strains u_r°, u_θ°, u_φ° and stresses σ_r°, $\tau_{r\varphi}^\circ$, $\tau_{r\theta}^\circ$ in the elastic medium, the Fourier transformants of which will be denoted by the number n in the subscript of a symbol, in accordance with (1.3), while the Fourier transformants of the combinations (2.7) of these strains and stresses will be denoted by $z_n^\circ, z_n^{\circ *}$, $\tau^\circ{}_n$, $\tau_n^{\circ *}$. Here, if instead of an incident elastic wave, the elastic medium is subjected to bulk forces, which vary with time as exp $(-i\omega_0 t)$ (this is the version of the problem which we will also have in mind), we will use the same notation for the corresponding amplitudes of the stresses and strains caused by this loading.

The stress and strain fields in the elastic medium due to this loading will be constructed in the form

$$u_{rn}(r,\theta)=\overset{\circ}{u}_{rn}(r,\theta)+\overset{1}{u}_{rn}(r,\theta),\quad u_{\theta n}=\overset{\circ}{u}_{\theta n}+\overset{1}{u}_{\theta n}\quad u_{\varphi n}=\overset{\circ}{u}_{\varphi n}+\overset{1}{u}_{\varphi n} \tag{3.1}$$

where u_{rn}°, $u_{\theta n}^\circ$, $u_{\varphi n}^\circ$ are the displacements due to the specified load of the elastic medium when there is no defect, and $u1_{rn}$, $\overset{1}{u}_{\theta n}$, $\overset{1}{u}_{\varphi}$ is the discontinuous solution constructed in Section 2. It was constructed for a defect of general form. In the case of the defect considered here in the form of an inclusion perfectly bonded to the elastic medium, the jumps in the strains in (2.6) should be zero, and hence

$$\langle u_{nk}\rangle,\ \langle \upsilon_{nk}\rangle,\ \langle w_{nk}\rangle,\ \langle z_{nk}\rangle,\ \langle z_{nk}^*\rangle=0$$

Consequently, we obtain from (2.11)

$$\begin{aligned}
&\langle \Psi_{1nk}\rangle=0,\ \ k(k+1)\langle \Psi_{1nk}'\rangle=\mu^{-1}\langle \tau_{nk}^*\rangle\\
&k(k+1)b^2\ \langle \Psi_{2nk}\rangle=\mu^{-1}\langle \tau_{nk}\rangle,\ \ b^2\langle \Phi_{nk}\rangle=-\mu^{-1}\langle \sigma_{rnk}\rangle\\
&b^2R\langle \Phi_{nk}'\rangle=-\mu^{-1}\langle \tau_{nk}\rangle,\qquad k(k+1)b^2R\langle \Psi_{2nk}'\rangle=\mu^{-1}[k(k+1)\langle \sigma_{rnk}\rangle-\langle \tau_{nk}\rangle]
\end{aligned} \tag{3.2}$$

Substituting these expressions into (2.12) and inverting the Legendre transformants, we obtain

$$\begin{aligned}
&\Psi_{1n}(r,\theta)=\frac{R^2}{\mu}\int_0^\omega \langle \tau_n^*\rangle \sin\tau K_*^n(\theta,\tau;\ r,R)d\tau\\
&\mu\Psi_{2n}(r,\theta)=\frac{R}{b^2}\left\{\int_0^\omega \langle \sigma_{rn}\rangle \sin\tau K_b^n(\theta,\tau;\ r,R)d\tau-\right.\\
&\left.-\int_0^\omega \langle \tau_n\rangle \sin\tau\left[K_*^n(\theta,\tau;\ r,R)+R\frac{\partial}{\partial R}K_b^n(\theta,\tau;r,R)\right]d\tau\right\}\\
&\Phi_n(r,\theta)=\frac{R}{Mb^2}\int_0^\omega \sin\tau\left[r\frac{\partial}{\partial R}K_a^n(\theta,\tau;\ r,R)\langle \sigma_{rn}\rangle-K_a^n(\theta,\tau;\ r,R)\langle \tau_n\rangle\right]d\tau
\end{aligned} \tag{3.3}$$

The kernels K_a^n and K_b^n are found from (1.12) for $d=a$ and $d=b$, respectively, while the kernel K_*^n is obtained from the same formula but with $\Gamma_{d,k}$ replaced by $[k(k+1)]^{-1}\Gamma_{b,k}$. Hence, in the case under discussion in representation (3.1) the functions $\overset{1}{u}_{rn}$, $\overset{1}{u}_{\theta n}$, $\overset{1}{u}_{\varphi n}$ will be found from (2.2) and (3.3).

In order to obtain the equations for determining the unknown jumps $\langle\sigma_{rn}\rangle$, $\langle\tau_n\rangle$ and $\langle\tau_n^*\rangle$ we must specify the conditions on the defect (the inclusion). If it is assumed to be fixed, the following conditions must be satisfied

$$u_n(R-0,\theta)=\upsilon_n(R-0,\theta)=w_n(R-0,\theta)=0,\ \ 0\leqslant\theta\leqslant\omega$$

or, taking (2.7) into account, the conditions

$$u_n(R-0,\theta)=z_n(R-0,\theta)=z_n^*(R-0,\theta)=0,\ \ 0\leqslant\theta\leqslant\omega \tag{3.4}$$

Specifying these conditions using (2.9) and (3.1), we reduce the problem to the following system of integro-differential equations

$$R\Phi_n'(r,\theta)|_{r=R-0} + \nabla_n \Psi_{2n}(R-0,\theta) = -Ru_r^0(R,\theta)$$
$$\nabla_n\{\Phi_n(R-0,\theta) - [r^2\Psi_{2n}(r,\theta)]'|_{r=R-0}\} = Rz_n^0(R,\theta) \quad (3.5)$$

$$\nabla_n \Psi_{1n}(R-0,\theta) = -z_n^{\theta *}(R,\theta), \quad 0 \leq \theta \leq \omega \quad (3.6)$$

As can be seen, the last equation can be solved independently of the previous ones, while the first two must be solved simultaneously.

We will give further details of the proposed method as it applies to the case when the incident wave is a torsional wave.

4. THE PROBLEM OF THE DIFFRACTION OF AN ELASTIC TORSIONAL WAVE BY A THIN SPHERICAL INCLUSION

We will take the incident torsional wave in the form

$$u_{\varphi 0}^0 = Ar\sin\theta\, \bar{e}^{ibr\cos\theta}, \quad A = \text{const}, \quad u_{\theta 0}^0 \equiv u_{r0}^0 \equiv 0 \quad (4.1)$$

Since there is axial symmetry, we must put $n = 0$ in all the previous formulae. If we take into account the differential equation which the associated Legendre polynomials $P_k^{|n|}(\cos\theta)$ satisfy, it can be shown that the following equalities hold

$$\nabla_n P_k^{|n|}(\cos\theta) = k(k+1)P_k^{|n|}(\cos\theta)$$
$$\nabla_n K_*^n(\theta,\tau;\, r,R) = K_b^n(\theta,\tau;\, r,R) \quad (4.2)$$

We obtain the integral equation for the problem in question from Eq. (3.6) by using (2.7), (3.1), (3.3) and (4.2) and putting $n = 0$

$$\int_0^\omega \left[\frac{W_0(\operatorname{tg}^{1/2}\theta, \operatorname{tg}^{1/2}\tau)}{2\cos\frac{1}{2}\theta\cos\frac{1}{2}\tau} - \frac{D_0(\cos\theta,\cos\tau)}{2}\right]\chi(\tau)\sin\tau d\tau = f(\theta) \quad (4.3)$$

$$W_m(x,y) = \int_0^\infty J_m(tx)J_m(ty)dt, \quad f(\theta) = A\mu(i\xi\sin^2\theta - 2\cos\theta)e^{i\xi\cos\theta}, \quad 0 \leq \theta \leq \omega \quad (4.4)$$

$$D_m(x,y) = \sum_{k=m}^\infty \frac{P_k^m(x)P_k^m(y)\Delta_k(\xi)}{(k+m)![(k-m)!]^{-1}}, \quad \xi = bR, \quad m = 0,1,2,\ldots \quad (4.5)$$

where the unknown function, by (2.7) and (3.3), is

$$\chi(\theta) = \langle \dot{\tau}_0(R,\theta)\rangle = \frac{1}{\sin\theta} \quad \frac{d}{d\theta}\langle \tau_{r\varphi}(R,\theta)\rangle \quad (4.6)$$

When obtaining Eq. (4.3) we took into account, using (1.12) and (1.11), that

$$K_b^0(\theta,\tau;\, R-0,R) = \frac{1}{2R}\sum_{k=0}^\infty A_k(\xi)(2k+1)P_k(\cos\theta)P_k(\cos\tau)$$

and we also used representation (1.14) and the results obtained in [2] on the summation of series in associated Legendre functions.

If we make the following substitution in (4.3)

$$\operatorname{tg}\tfrac{1}{2}\theta = \beta x, \quad \operatorname{tg}\tfrac{1}{2}\tau = \beta y, \quad \beta = \operatorname{tg}\tfrac{1}{2}\omega, \quad X(y) = \frac{\chi(2\operatorname{arctg}\beta y)}{(1+\beta^2y^2)^{3/2}} \quad (4.7)$$

$$[(1+r^2)(1+s^2)]^{1/2} R_0(r,s) = D_0\left(\frac{1-r^2}{1+r^2}, \frac{1-s^2}{1+s^2}\right), \quad F(x) = \frac{f(2\operatorname{arctg}\beta x)}{2\beta\sqrt{1+\beta^2x^2}}$$

we obtain the well-known equation [1]

$$\int_0^1 [W_0(x,y) - \beta R_0(\beta x, \beta y)]\frac{X(y)}{y}dy = F(x), \quad 0 \leq x \leq 1 \tag{4.8}$$

which has been solved by many researchers, but in the class of integrable functions. However, here we must solve it in the class of functions having non-integrable singularities, since the required function in Eq. (4.3), from which (4.8) is obtained, is, by (4.6), the derivative with respect to θ of the stress jump $\tau_{r\varphi}(r, \theta)$. This jump at the edge of the inclusion $\theta = \omega$ has a root singularity and, consequently, its derivative with respect to θ will have a power singularity with index –3/2.

Equation (4.8) was solved previously in [8] in the class of such functions. Below we propose another, more direct method based on a new eigenvalue relation

$$\int_0^1 \frac{W_m(x,y)P_k^{m,-3/2}(1-2y^2)}{y^{-m-1}(1-y^2)^{3/2}}dy = \frac{\Gamma(m+k+1/2)\Gamma(k-1/2)}{2k!}\times$$
$$\times\begin{cases} x^m P_{k-1}^{m,1/2}(1-2x^2)\Gamma^{-1}(k+m), & 0 \leq x \leq 1 \\ \dfrac{F(m+k+1/2, k+1/2;\ 2k+m+1/2;\ x^{-2})}{\Gamma(1/2-k)\Gamma(2k+m+1/2)x^{2k+m+1}}, & 1 < x < \infty, \quad m = 0,1,2,\ldots \end{cases} \tag{4.9}$$

Here we must assume $P_{k-1}^{m,1/2}(1-2x^2) \equiv 0$ for $k = 0$ and the integral must be understood in the generalized (regularized [9, 1]) sense. Relation (4.9) is proved using the scheme developed in [10].

Having the eigenvalue relation (4.9), to solve integral equation (4.6) we can use the orthogonal polynomial method [1], i.e. we search for a solution in the form of a series

$$X(y) = \sum_{k=0}^{\infty} X_k(1-y^2)^{-3/2}P_k^{0,-3/2}(1-2y^2) \tag{4.10}$$

After substituting this series into (4.8) we must take into account that, by the considerations presented in [8], the following condition must be satisfied

$$\int_1^{\infty} J^0(x)xdx < \infty, \quad J^0(x) = \int_0^1 W_0(x,y)yX(y)dy$$

An analysis of the integral $J^0(x)$ using relation (4.9) as $x \to \infty$ and $x \to 1+0$ shows that only the first term of series (4.10) does not satisfy this condition and it must therefore be discarded, i.e. series (4.10) must be taken in the form

$$X(y) = \sum_{l=0}^{\infty} X_{l+1}(1-y^2)^{-3/2}P_{l+1}^{0,-3/2}(1-2y^2) \tag{4.11}$$

Carrying out the standard scheme of the orthogonal polynomial method [1], we reduce the integral equation (4.8) to an infinite system

$$Y_j - \sum_{l=0}^{\infty} \frac{Y_l d_{jl}}{\sqrt{N_j N_l \sigma_j \sigma_l}} = \frac{F_j}{\sqrt{N_j \sigma_j}}, \quad j = 0,1,\ldots \tag{4.12}$$

Here

$$Y_j = \sqrt{N_j\sigma_j}X_{j+1}, \quad 2\sigma_l = \Gamma(l+1/2)\Gamma(l+3/2)[l!(l+1)!]^{-1} \tag{4.13}$$
$$N_l^{\alpha,\beta} = \int_0^1 \left|P_l^{\alpha,\beta}(1-2y^2)\right|^2 y^{2\alpha+1}(1-y^2)^{\beta}dy =$$
$$= \Gamma(\alpha+l+1)\Gamma(\beta+l+1)[2(\alpha+\beta+2l+1)l!\Gamma(\alpha+\beta+l+1)]^{-1}, \quad N_l^{0,1/2} \equiv N_l$$

$$F_j = \int_0^1 \frac{xF(x)}{(1-x^2)^{-1/2}} P_j^{0,1/2}(1-2x^2)dx = F_j^{0,1/2} \tag{4.14}$$

$$d_{jl} = \int_0^1 \int_0^1 \frac{P_j^{0,1/2}(1-2x^2)P_{l+1}^{0,-3/2}(1-2y^2)\beta R_0(\beta x, \beta y)dxdy}{(1-x^2)^{-1/2}(1-y^2)^{3/2}(xy)^{-1}} \tag{4.15}$$

The last integral must be understood in the generalized (regularized) sense. Using (4.7) and (4.5) it can be reduced to the form

$$d_{jl} = \sum_{k=0}^{\infty} \Delta_k(\xi) I^{k,j}_{1/2,0,0}(\beta) I^{k,l+1}_{-3/2,0,0}(\beta) \tag{4.16}$$

$$I^{k,n}_{q,p,r}(\beta) = \int_0^1 P_k\left(\frac{1-\beta^2x^2}{1+\beta^2x^2}\right) \frac{P_n^{0,q}(1-2x^2)xdx}{(\beta^2x^2)^{-r}(1+\beta^2x^2)^{1/2+p}(1-x^2)^q} =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i(-i-r)_n \beta^{2(i+r)} c_i}{(i+r)!^{-1} n!\Gamma(2+n+i+r+q)}$$

$$c_i = \sum_{m=0}^{i} \left(\frac{(-k)_m}{m!}\right)^2 \frac{(1/2+k+p)_{i-m}}{(i-m)!}$$

When evaluating the last integral an expansion in a Maclaurin series in βx is carried out, which depends on this variable, using formulae 8.962(1) and 9.121(1) from [5], and formula 7.391(4) from [5] is then used.

The coefficients $\Delta_k(\xi)$ must be calculated from (1.14) using well-known power expansions [5] for the Bessel and Struve functions, and also the tabulated integrals 3.632(17) from [5]. The coefficients F_j must be calculated using the well-known expansion [11]

$$e^{-i\xi\cos\theta} = \sum_{k=0}^{\infty} P_k(\cos\theta)(2k+1)e^{k\pi i/2} J_{k+1/2}(\xi)$$

We then obtain, using (4.14), (4.7) and (4.4)

$$2^{3/2}\beta F_j = A\mu\sqrt{\pi}\sum_{k=0}^{\infty}(2k+1)e^{k\pi i/2}J_{k+1/2}(\xi)[4i\xi I^{k,j}_{1/2,2,1}(\beta) - 2I^{k,j}_{1/2,1,0}(\beta) + 2I^{k,j}_{1/2,1,1}(\beta)] \tag{4.17}$$

We will solve the infinite system (4.13) approximately by the reduction method, the basis of which will be given below.

If we assume the inclusion to be fixed, an important characteristic is the reactive torque

$$M_0 = 2\pi R^3 \int_0^{\omega} \langle \tau_{r\varphi}(R,\theta)\rangle \sin^2\theta d\theta \tag{4.18}$$

From (4.6) we can establish the equation

$$\sin\theta\langle\tau_{r\varphi}(R,\theta)\rangle = \int_0^{\vartheta} \sin\tau\chi(\tau)d\tau$$

Substituting this expression into (4.18), integrating by parts, replacing the variables (4.7) and using (4.11), taking (4.13) into account, we obtain the formula

$$M_0 = 16\pi R^3\beta^4 \sum_{l=0}^{\infty} \frac{(l+1)^{-1}}{\sqrt{N_l\sigma_l}} \sum_{j=l}^{\infty} \frac{(-1)^{l+j}(j+1)!^2\beta^{2l}}{(j-l)!(1/2+l)_{j+2}} \tag{4.19}$$

We will now consider the case when the inclusion is not fixed and can rotate due to the action of the incident wave. We will denote the amplitude of the angle of rotation by α. In this case the conditions imposed on the defect (1.2) must be changed and written as follows: $z_0^{1*}(R,\theta) + z_0^{0*}(R,\theta) = 2\alpha R\cos\theta$. Substituting z_0 from (2.8) here, as previously, we obtain integral equation (4.3), where instead of $f^{(\theta)}$

we must take the sum $f(\theta) + \alpha g(\theta), g(\theta) = 2\mu \cos \theta$, and, correspondingly, we obtain the infinite system (4.13) in which instead of F_j, we must take the sum $F_j + \alpha G_j$, where

$$G_j = \frac{\mu}{\beta} \int_0^1 \frac{(1-\beta^2 x^2) P_j^{0,1/2}(1-2x^2) x dx}{(1-x^2)^{-1/2}(1+\beta^2 x^2)^{3/2}} = \frac{\mu}{2\beta j!} \sum_{k=0}^{\infty} \frac{(-1)^k (3/2)_k}{\beta^{-2k} k!} (e_k - \beta^2 e_{k+1}), \quad e_k = \frac{(-k)_j k!}{(3/2 + j)_{k+1}} \tag{4.20}$$

The structure of the solution of the infinite system will be similar, i.e. $Y_j = Y_j^0 + \alpha Y_j^\alpha$, where Y_j^0 is the solution of the previous infinite system (4.13), while Y_j^α is the solution of system (4.13), where we must take G_j from (4.20) instead of the coefficients F_j.

The reactive torque M for the inclusion will be found from the formula $M = M_0 + \alpha M_\alpha$, where M_0 and M_α are calculated from (4.19) with Y_1 replaced by Y_l^0 and Y_l^α, respectively. Using d'Alembert's principle, we obtain the following expression for the amplitude α of the terminal oscillations of the inclusion

$$\alpha = (M_\omega - M_0) M_\alpha^{-1} \tag{4.21}$$

where $M_\omega = -\pi\omega^2{}_0\rho_0\delta R^3$ ($\omega - \sin\omega\cos\omega$) is the torque due to the inertial forces, δ is the thickness of the inclusion and ρ_0 is its density.

5. THE BASIS OF THE REDUCTION METHOD AND NUMERICAL RESULTS

According to the well-known results in [12], the reduction method for solving infinite system (4.12) will be usable if we can prove that the following series converge

$$S_1 = \sum_{j,l=0}^{\infty} \frac{|d_{jl}|^2}{N_j N_l \sigma_j \sigma_l}, \quad S_2 = \sum_{j=0}^{\infty} \frac{|F_j|^2}{N_j \sigma_j} = \sum_{j=0}^{\infty} \frac{|F_j^{0,1/2}|^2}{N_j \sigma_j} \tag{5.1}$$

To prove this we will introduce the Fourier coefficient in the expansion of the integrable function $h(x)$ in Jacobi polynomials $P_n^{\alpha,\beta}(1-2x^2)$ in the interval [0, 1]

$$h_n^{\alpha,\beta} = \int_0^1 h(x) P_n^{\alpha,\beta}(1-2x^2) x^{2\alpha+1}(1-x^2)^\beta dx \tag{5.2}$$

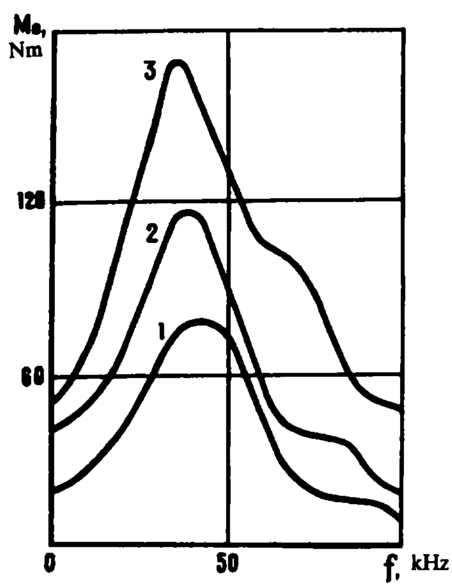

Fig. 1.

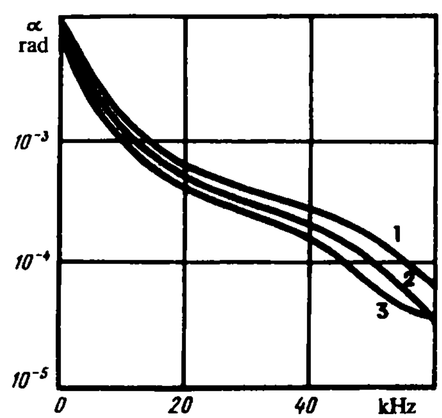

Fig. 2.

If the function is continuously differentiable in the interval [0, 1], we will denote the Fourier coefficient in this expansion of the function $x^{-1}h'(x)$ by $\tilde{h}_n^{\alpha,\beta}$. If we now carry out integration by parts in (5.2), taking into account the relation

$$\int_0^x \frac{P_n^{\alpha,\beta}(1-2\xi^2)\xi d\xi}{[\xi^{2\alpha}(1-\xi^2)^{\beta}]^{-1}} = \frac{x^{2\alpha+2}P_{n-1}^{\alpha+1,\beta+1}(1-2x^2)}{2n(1-x^2)^{-\beta-1}}, \quad n>0$$

which follows from A6.12 of [1], we can establish that

$$h_n^{\alpha,\beta} = -(2n)^{-1}\tilde{h}_{n-1}^{\alpha+1,\beta+1} \tag{5.3}$$

These operations in obtaining (5.3) are justified if Re $(\alpha, \beta) > -1$. They are also justified when $\beta = -3/2$ by virtue of considerations presented in [13, 1].

We will prove the convergence of the second series in (5.1).

Taking into account the fact that, by virtue of (5.3), we can write the relation $F_j^{0,1/2} = (2j)^{-1}\tilde{F}_{j-1}^{1,3/2}$, this series can be rewritten in the form

$$S_2 = \frac{F_0^{0,1/2}}{N_0\sigma_0} + \sum_{k=0}^{\infty} \frac{|\tilde{F}_k^{1,3/2}|^2}{2(k+1)N_{k+1}\sigma_{k+1}}$$

The convergence of the last series can be established if we use the Parseval equation [14]

$$\sum_{k=0}^{\infty} \frac{|\tilde{F}_k^{1,3/2}|^2}{N_k^{1,3/2}} = \int_0^1 \frac{F'(x)x^3dx}{x(1-x^2)^{-3/2}}$$

The integral on the right exists by virtue of (4.7) and (4.4), which define the function $F(x)$.

To prove the convergence of series S_1 in (5.1), we will represent it in the form of two series

$$S_1 = S_1^0 + S_1^1, \quad S_1^0 = \sum_{l=0}^{\infty} \frac{|d_{0l}|^2\sigma_l^{-1}}{N_0\sigma_0 N_l}$$
$$S_1^1 = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty} \frac{|d_{jl}|^2\sigma_l^{-1}}{N_j N_l \sigma_j} \tag{5.4}$$

We introduce the following notation

$$\tilde{R}_0(x,y) = \frac{\beta}{xy}\frac{\partial^2 R_0(\beta x,\beta y)}{\partial x\partial y}, \quad \tilde{g}_l^{1,-1/2}(x) = \int_0^1 \frac{1}{y}\frac{\partial R_0(\beta x,\beta y)P_l^{1,-1/2}(1-2y^2)dy}{\partial y\sqrt{1-y^2}\,y^{-3}}$$
$$\tilde{d}_{kl} = \int_0^1\int_0^1 \tilde{R}_0(x,y)\frac{P_k^{1,3/2}(1-2x^2)P_l^{1,-1/2}(1-2y^2)dxdy}{(1-x^2)^{-3/2}\sqrt{1-y^2}\,x^{-3}y^{-3}} \tag{5.5}$$

Using (4.7), (4.5) and (5.3) we have the equations

$$R_0(\beta x,\beta y) = \sum_{k=0}^{\infty} \Delta_k(\xi)P_k\left(\frac{1-\beta^2x^2}{1+\beta^2x^2}\right)P_k\left(\frac{1-\beta^2y^2}{1+\beta^2y^2}\right)$$
$$d_{jl} = \frac{d_{j-1,l}}{2(l+1)2j} \tag{5.6}$$

Using the above notation we obtain the Parseval equation

$$\int_0^1\int_0^1 \left|\frac{1}{y}\frac{\partial R_0(\beta x,\beta y)}{\partial y}\right|^2 \frac{y^3dxdy}{\sqrt{1-y^2}} = \sum_{l=0}^{\infty}(N_l^{1,-1/2})^{-1}\int_0^1 \left|\tilde{g}_l^{1,-1/2}(x)\right|^2 dx,$$
$$\int_0^1\int_0^1 \left|\tilde{R}_0(x,y)\right|^2\sqrt{\frac{1-y^2}{1-x^2}}x^3y^3dxdy = \sum_{j,k=0}^{\infty}\frac{|\tilde{d}_{kl}|^2}{N_k^{1,-1/2}N_l^{1,1/2}} \tag{5.7}$$

where the double integrals in (5.7) exist. This follows from representation (5.6) for $R_0(\beta x, \beta y)$, from which it can be seen that the function $R_0(\beta x, \beta y)$ will be continuously differentiable if $\Delta_k(\xi) = O(k^{-m})$ and $k \to \infty$ and $m > 4$. We can obtain this asymptotic form for $\Delta_k(\xi)$ if we carry out integration by parts the required number of times in (1.14) and bear in mind that $Q_0(z)$ is an integral function.

In order to show that series S_1^1 from (5.4) converge, it is sufficient to put $j = k+1$ in its representation and use the second equations in (5.6) and (5.7). To prove that series S_1^0 from (5.4) converges, using the representation for d_{jl} from (4.15) and the Cauchy–Bunyakovskii inequality, we can establish the relation

$$|d_{0l}|^2 \leqslant \frac{1}{30(l+1)^2} \int_0^1 |\bar{g}_l^{1,-1/2}(x)|^2 \, dx$$

which, in combination with the first equation of (5.7), guarantees that the series $S^0{}_1$ converges.

When calculating the reactive torque from (4.19) and the amplitude of the torsional oscillations of the inclusion from (4.21), the input parameters were fixed as follows [4]: the inclusion was made of steel of thickness $\delta = 5 \times 10^{-4}$ m, radius $R = 0.02$ m, and density $\rho_0 = 7900$ kg/m^3, while the material of the elastic medium was calcite with a shear wave velocity $c_2 = 1113$ m/s and a Lamé parameter $\mu = 3.58 \times 10^9$ MPa, and the amplitude of the incident wave $A = 0.01$ m.

In Figs 1 and 2 we show graphs of the reactive torque M_0 and the logarithm of the amplitude α of the oscillations of the inclusion as a function of $f = \omega_0/(2\pi)$; curve 1 corresponds to $\omega \approx 36°$ and $\beta = 0.32$.

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